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BACHELOR THESIS

Incentive Schemes Under O-Ring Production Functions

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Version as of June 20, 2014

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"Unfortunately, like many people who are instinctively bad at something, the Archchancellor prided himself on how good at it he was. Ridcully was to management what King Herrod was to the Bethlehem Playgroup Association. His mental approach to it could be visualized as a sort of business flowchart with, at the top, a circle entitled 'Me, who does the telling' and, connected below it by a line, a large circle entitled 'Everyone else'. Until now this had worked quite well, because, although Ridcully was an impossible manager, the University was impossible to manage and so everything worked seamlessly".

Terry Pratchett, *The Last Continent*

1 Introduction

Certainly, not every manager is as impossible as Archchancellor Ridcully of the Unseen University. However, the two-circle flowchart might still be a lucid illustration of how sometimes the supervision works. One particular way in which managers are often criticized is their low level of engagement into the production process. Moreover, the main accusation made against the managers is usually not them having their "hands off" the actual "doing", but rather the lack of general insight about the activities of their subordinates. While not being aware of how complex the tasks are or how mundane the job is for sure does not help to earn respect of the junior staff, arguably the most irritating situation is the one in which manager has no clue about the efforts put in by the team. This paper will focus precisely on the analysis of such cases.

While almost anybody would love to see his contribution appreciated by the superior, any economist would point out that management's lack of insight with regards to staff performance is a two-edged sword: it creates a auspicious environment for moral hazard. Moral hazard is a form of information asymmetry - when manager (or as she is more commonly referred in the literature - the principal[1]) is not able to observe the efforts of the workers (or as they are also called, agents). Then, the famous principal-agent problem arises - agent has incentives to "shirk", or extort low level of effort, which in turn might undermine the outcome of the production. This is especially true when the production has some inherent random component that might influence its results - then production failure could not be attributed to the low level of efforts with certainty. This particular type of models leads to an interesting conclusion - under certain types of risk-preferences, efficient co-insurance schemes will exist. A specifically illuminating case of such a situation could be in setting proposed by Lazear & Rosen [8], who suggest that rank-based agent compensation might be particulary efficient compared to piece rate under significant common production shocks.

Principal faces even greater issues when there are multiple agents required for production of the good, and there exists complementarity between their inputs (whether they are exogenously given,

[1]Please note that this paper will consider principal to be female, while her subordinates to be males only for the sake of convenience and being politically correct.

like skill, or endogenous, like efforts). On of the first treatises of this phenomena is the seminal paper by Alchian & Demsetz [2], in which they discuss the implications of such complementarity for the firm structure[2]. Another paper that discusses the related issue is Holmstrom [6], which investigates how teams without a principal will be inefficient in terms of overall production due to the presence of externalities (worker who gets a share $s < 1$ of the output Q created by her efforts e will choose to extort efforts $s\frac{\partial Q}{\partial e} = \frac{\partial \psi}{\partial e}$, where $\psi(e)$ is the cost of efforts). Note, however, the socially optimal level of efforts would be greater, as it would be the solution of a slightly different equation, $\frac{\partial Q}{\partial e} = \frac{\partial \psi}{\partial e}$. Holmstrom himself suggests that one solution to such a problem might be a delegation of the monitoring duties to a third party - effectively, introducing the role of the principal.

However, our paper will contribute to the stream of literature that aims to investigate what might be the optimal incentive schedule in a very particular setting - under O-ring production functions. Firstly introduced by Kremer [7] in his attempt to explain differences in wage level both between and within countries and firms, this function incorporates the two properties discussed above - it has both input complementarity and inherent uncertainty about the production results. Specifically, output y is considered to be binary - it is either 1 if the production process is successful, and 0 if it is not. The product itself, in turn, would only be produced if **all** the individual tasks x_j required would be successfully performed by the agents. Outcome of the each task x_j is also binary, and the probability of its success is some $0 < q_j < 1$, where q is exogenously given input, like skill of the agent. Therefore, the overall production function would be given by

$$y = \prod_{j}^{N} x_j, \tag{1}$$

where N is the number of tasks. Assuming that all the task are performed simultaneously and that results of individual production are independent, we can state that the expected level of production (and also the probability of success) would be given by

$$E(y) = E(\prod_{j}^{N} x_j) = (\prod_{j}^{N} q_j). \tag{2}$$

From the very nature of the O-ring production function we can state that there inherent risks that could create grounds for moral hazard. Moreover, is is also clear that input complementarity in the sense of Alchian & Demsetz [2]is present with respect to q_j is present:

$$\frac{\partial^2 y}{\partial q_i \partial q_j} = \prod_{l \neq j,i}^{N} q_l > 0 \forall\ i \neq j. \tag{3}$$

[2]Besides being famous for the introduction of the very important notion of input complementarity (described as $\frac{\partial^2 Z}{\partial X_1 \partial X_2} \neq 0$, where Z is the output and X_1 and X_2 are the inputs), this paper also reminds us of the merry old times when it was possible to publish a paper in *American Economic Review* with just one formula in it (provided here).

After introducing the function Kremer derives the wage schedules for different levels of q under zero profit condition, and how workers would be matched within countries and firms. He focuses his analysis on the effects of input complementarity, later on taking to the extreme when investigating the case of sequential production, when further tasks require successful completion of all previous tasks. Kremer also discusses how agents should be allocated to tasks, and what the optimal level of the good complexity the principal should choose.

However, as in his model q_j's are given exogenously and workers do not make a decision about the level of efforts, there is no room for workers to "shirk". And that leaves us with an interesting gap in the literature - what is the optimal contract structure under O-ring production function under moral hazard? Answering this question is the main goal of this paper.

While currently there is no literature regarding this subject of this paper, for sure there are others working on it. The most relevant research in the field would be the working paper by Ghatak & Karaivanov [5]. Authors study the implications for contracts between two agents under a modification of O-ring production function with efforts. However, their form is complementary only in terms of skills, q_j, and not in terms of efforts, e_j:

$$y = \eta q_1 q_2 + q_1 e_1 + q_2 e_2 + \varepsilon, \tag{4}$$

where e_j is the ability of j-th agent and ε is some random shock. Moreover, note that the output produced is not binary - another important distinction from our paper. Under no effort complementarity the optimal choice of effort by agent does not depend on choice of others, but is lower than the socially optimal level (a result closely resembling the one obtained by Holmstrom [6]). As we would show later, effort complementarity and O-ring production function significantly impacts contract structure.

This paper will investigate the incentive schemes under O-ring production functions. Firstly, we would investigate model with only one agent with particular emphasis on the implications of the risk preferences of the agent. The models discussed would be without endogenous efforts, as well as with endogenous unobservable and observable efforts. We would show that under risk-neutrality there is little difference between those three cases.

The next section will discuss models where two agents interact, and explore how the efforts complementarity impacts the options available to the principal - she is forced to pay high enough bonuses to make agents to extort non-zero efforts. We will also briefly consider case with observable individual production results, and how enforceable minimal efforts requirement might effect principal's options in terms of bonuses. Section considers a generalization of our model to the N agent case.

2 One agent models of contracts under O-ring production function

2.1 Model with one agent and no endogenous efforts

Consider a principal who requires only one agent to produce some output y, which is equal to unity if the good is produced successfully and zero if the production fails. The production function for y is similar to the Kremer's O-ring function [7], but is contingent only on the successful completion of one task - thus, $y = x$, where x is the individual performance of the worker[3]. Principal sells the good for some price p (exogenously given), and has to pay wage w to the worker. Thus, principal's profits are given by $Pr = py - w$. Note, however, that principal is not necessarily risk-neutral, and could be maximising some function $V(Pr)$, such that $\frac{\partial V}{\partial Pr} > 0$, but $\frac{\partial^2 V}{\partial Pr^2} \neq 0$[4]. Moreover, we assume that the reserve utility of the principal corresponds to the $V(0)$ - thus, $Pr \geq 0$ is the individual rationality constraint for the principal.

As for the agent, his individual output is a Bernoulli-distributed random variable x, and takes two possible values: $x = 1$ if the production is successful (this outcome has a probability of q) and $x = 0$ if the production fails (with probability of $1 - q$). In this setting, $0 \leq q \leq 1$ is a measure of how good the worker is in producing the output, and might represent skill, experience, education, or even individual luck. The important feature of q is that it is exogenously given at the time of the production, and none of the parties can influence it[5]. Moreover, both principal and agent know the value of q prior to the output production - it means that no adverse selection is present in the model. Agent maximizes his utility of $U(w)$ such that $\frac{\partial U}{\partial w} > 0$ and $\frac{\partial^2 U}{\partial w^2} \geq 0$[6], and also has reservation utility of $\bar{U}(q)$. It is reasonable to assume that the reservation utility, which measures how good are the outside options of the worker, would positively depend on his skill, education, etc.

We assume that agent is "pre-assigned" to the principal in a sense that principal can contract only with agent with given q. The model itself has two stages: firstly, the contract is designed by principal and offered to the agent, who can either consent or decline. Secondly, the production process occurs. The optimal contract would heavily depend on risk preferences of both parties.

The principal considers setting wage schedule of $w = \alpha + x\beta$ for the worker[7]. In this schedule,

[3] Even though for now such notation seems excessive (we have overall output y equal to the individual output x), we would require the distinction later on when we will consider models with multiple workers required for the production.

[4] However, examples of risk-loving principals are scarce, and we would not consider such instances in our paper.

[5] Later this assumption will also be relaxed.

[6] Even though different assumptions about agent's risk preferences could be made, throughout the paper we would only consider risk-neutral or risk-averse agents.

[7] Please note that analysis in this and the following sections closely follows one presented in Bolton et al. [3], section 4.1 (130-137).

α stands for salary, or fixed part of the wage, which is paid to the worker irrespectively of the production result, while β represents a 'bonus" in case the good was successfully produced.[8] Principal sets both α and β in order to maximize her own expected utility, given by

$$E(V(Pr)) = qV(p-\alpha-\beta) + (1-q)V(-\alpha)). \tag{5}$$

The only constraint here is the individual rationality constraint for the worker, and as he is also risk neutral, it is given by $E(U(w)) \geq \bar{U}(q)$, or

$$qU(\alpha+\beta) + (1-q)U(\alpha) \geq \bar{U}(q). \tag{6}$$

Thus, combining (5) and (6), we obtain the maximization problem for the principal:

$$\begin{cases} qV(p-\alpha-\beta) + (1-q)V(-\alpha) \to \max\limits_{\alpha,\,\beta}; \\ \text{s.t. } qU(\alpha+\beta) + (1-q)U(\alpha) \geq \bar{U}(q). \end{cases}$$

Solving this problem would yield the following first-order conditions[9]:

$$qV'(p-\alpha-\beta) + (1-q)V'(-\alpha) = \lambda qU'(\alpha+\beta) + \lambda(1-q)U'(\alpha) \text{ and}$$

$$(1-q)V'(-\alpha) = \lambda(1-q)U'(\alpha).$$

Therefore, at optimum the following equality should hold:

$$\frac{V'(p-\alpha-\beta)}{U'(\alpha+\beta)} = \frac{V'(-\alpha)}{U'(\alpha)} = \lambda^{10}. \tag{7}$$

Note that up to this point we have not used any assumptions regarding risk preferences of both agent and principal - the optimum condition obtained is rather general in this respect.

If both agents are risk-neutral, it implies that $\frac{\partial^2 V}{\partial Pr^2} = 0$ and $\frac{\partial^2 U}{\partial w^2} = 0$, or, in terms of first-order derivatives, that both $\frac{\partial V}{\partial Pr}$ and $\frac{\partial U}{\partial w}$ are constant and positive[11]. That would mean that condition (7) would be satisfied irrespective of particular values of α and β chosen by the principal.

If the principal is risk-neutral but the worker is risk averse (thus, $\frac{\partial V}{\partial Pr}$ is constant while $\frac{\partial U}{\partial w}$ is not), numerators in both fractions in (7) are equal. For equality to hold, denominators must also be equal, thus, $\beta^* = 0$ and all the risk is taken by the principal. Note that if the risk preferences were reversed, for (7) to hold, $\beta^* = p$ and all the risk should be taken by the worker. The optimal

[8] Note that such a wage schedule allows to set risk-free wage for worker (if $\beta = 0$), or risk-free profits for principal (if $\beta = p$). Moreover, note that there are no constraints on whether α or β should have a specific sign.

[9] Note that the individual rationality constraint would be binding, as principal has no incentive to pay the worker more that he agrees to work for.

[10] Note that the condition obtained is the so-called "Borch rule", obtained by Borch [4] in a far more compicalted analysis of risk transfers in reinsurance market.

[11] Though not necessarily equal!

level of fixed part of the wage schedule, α^*, would be set by principal at minimal level sufficient to satisfy the individual rationality constraint of the worker (6).

If both parties were risk-averse, some intermediate solution would be present (depending on the particular forms of $V(Pr)$ and $U(w)$ functions)[12].

In any case, there is also a decision to be made by principal - whether the profits left would be sufficient to cover her reserve utility of $V(0)$ if the $w = \alpha^* + x\beta^*$ wage schedule would be offered to the worker. The principal compares $E(V(Pr)) = qV(p - \alpha^* - \beta^*) + (1-q)V(-\alpha^*)$ with $V(0)$, and if the former is greater, the optimal contract found is beneficial for the principal and thus would be signed.

In this very simple version of the model, "bonus" in case of successful production serves only as means for risk transfer between the agents, acting as a form of insurance inside the labour contract.

The main conclusion to be drawn from this subsection is that the optimal contract is significantly influenced by the risk preferences of the agents. Therefore, if we were to expand the model to incorporate other concepts in order to investigate their influence on the outcome, we should always compare results under the same risk preferences. However, as the model is developed to include new features, it becomes increasingly difficult to maintain the level of generality demonstrated in this subsection. Even though we would aim to conduct out analysis in general form, in the end we would almost always rely on risk-neutral agents, in particular with $V(Pr) = Pr$ for principal and $U(w) = aw$.

2.1a Model with risk-neutral principal and agent

In this section we would examine a particular case of the model described above - namely, with both principal and agent being risk-neutral. Even though we already know from the general analysis performed earlier that the contract structure is irrelevant in this case, we would need those results to serve as a benchmark for further analysis.

We assume linear utility for the principal to be $V(Pr) = Pr$, and linear utility for the agent to be $U(w) = aw$, where a is a positive constant[13].

As we know form the general analysis performed earlier, size of β, or success fee, is irrelevant and only serves as means to transfer risk from principal to the agent. Assuming that principal decides on some β^*, he would set α^* in order to satisfy agent individual rationality constraint, (6),

[12] Note that $\frac{p-\beta}{p}$ could be considered as a measure for the "share" of risk taken by principal. If both agents are risk-neural, any value of this ratio is optimal. In the second case considered this ratio would be equal to 1, in third - to zero, while under both agents being risk-averse it would be somewhere between 0 and 1.

[13] Event though right now this coefficient seems meaningless, it will be required later on to represent relative importance of income to the agent compared to costs of effort.

which we would rewrite given our linearity assumption for agent's utility function:

$$E(U) = qa(\alpha^* + \beta^*) + (1-q)a\alpha^* \geq \bar{U}. \tag{8}$$

As there is no incentive for the principal to set compensation scheme for the agent for his expected utility to be above reservation level, so (8) is in fact binding. Thus, given the β^*, principal sets $\alpha^* = \frac{\bar{U}}{a} - q\beta^*$. Therefore, expected profit (and utility) for the principal becomes $E(Pr) = q(p - \beta^*) - \alpha^* = pq - \frac{\bar{U}}{a}$.

Note that as was demonstrated earlier, payoffs of both principal and agent are independent of the value of β^* chosen by the principal. As for effects of different parameters, they are quite intuitive: p increases expected profits as the the successful production means greater revenue, q increases chance of the product being produced and thus the chances of getting the revenues and thus profits, higher $\bar{U}$ means that worker requires greater salary to agree to work, and higher a stands for greater importance of income in utility - lower wage could be offered.

2.2 Model with one agent and endogenous unobservable efforts

So far the model has not included any actions on the part of the agent (besides his choice whether to accept or to reject the contract offer made by the principal). The next step would be to introduce efforts into the model - ability of the agent to influence the probability of successful production. To do so, we now state that the individual output, x, is now distributed as a Bernoulli random variable with the probability of success given by qe, where $0 \leq e \leq 1$ is the level of efforts the worker chooses after the contract is signed. Therefore, if he devotes maximal efforts towards production, e would be equal to 1, while her probability of success would reach its potential value of q[14].

Introduction of effort should bring some trade-off for the worker, and thus should be a variable determining his level of utility: $U(w, e)$ such that $\frac{\partial U}{\partial e} < 0$, as effort is assumed to be costly for the agent.

For now effort e is considered to be observable by the principal. Note that paying a fixed wage no matter what is no more a option - as agent faces only costs of effort and no benefits, optimal ϵ would be equal to zero. Therefore, principal should construct some incentive scheme to motivate agent to work harder. One example of such schedule could be $w = \alpha + x\beta$, the same one we used earlier. Assuming the contract has been signed, the agent maximizes her expected utility with respect to effort level:

$$qeU(\alpha + \beta, e) + (1 - qe)U(\alpha, e) \to \max_e. \tag{9}$$

[14]Note that effort is introduced into production function *multiplicatively* - in contrast with *additive* effort, it guarantees that probability of success is always between 0 and 1. It also allows for meaningful interpretation of limits of this probability, with q the "ceiling" that bounds the prospective efficiency of production from above.

The respective first-order condition will be:

$$qU(\alpha+\beta,e)+qe\frac{\partial U(\alpha+\beta,e)}{\partial e}+\frac{\partial U(\alpha,e)}{\partial e}-qU(\alpha,e)-qe\frac{\partial U(\alpha,e)}{\partial e}=0. \tag{10}$$

As equation (10) is rather complex and no straightforward conclusions could be made from it, additional simplifying assumptions should be made. For instance, a separable in wage and effort utility function is common in related literature (see, for instance, [1], [3], [5], [6], [8]):

$$U(w,e)=u(w)-\psi(e), \tag{11}$$

where $\psi(e)$ stands for costs of effort (and is increasing in e)[15].

One immediate conclusion to made from (11) is that $\frac{\partial U(w,e)}{\partial e}$ is independent of level of w, and thus the (10) could be simplified to

$$q(U(\alpha+\beta,e)-U(\alpha,e))+\frac{\partial U(\alpha,e)}{\partial e}=0, \text{ or}$$

$$q(u(\alpha+\beta)-u(\alpha))=\psi'(e). \tag{12}$$

In order to simplify notation, we would label $u(\alpha+\beta)-u(\alpha)$ as du. In fact, it represents a utility "premium" or differential caused by the bonus payment β being added to agent's compensation in case of successful production of output.

Under an assumption that $\psi''(e)>0$, what implies increasing marginal cost of efforts, two important conclusions are to be made from equation (12). Firstly, size of "success" bonus β induces the agent to work harder (leads to greater optima e). Secondly, the greater is the skill of the agent, q, the lower will be the effort level e.

To simplify the calculations further, a specific effort cost function would be assumed (also frequently used in literature - see, for instance, [1], [3], [5]): $\psi(e)=\frac{e^2}{2}$. Under this assumption, (12) becomes

$$e=qdu,$$

and thus agent's optimal choice of effort would be:

$$e^*=\begin{cases}1, \text{ if } qdu>1;\\ qdu, \text{ if } 0\le qdu\le 1;\\ 0, \text{ if } qdu<0.\end{cases} \tag{13}$$

This optimal choice of the agent would be used by the principal for determining the optimal wage schedule in order to maximize her utility. Note that setting $\beta<0$ has no economic sense - it would make worker to choose $e^*=0$, and that means that principal will make no revenue and

[15] Note that as effort is chosen prior to the production process and thus is the same irrespective of whether output would be produced, $u(w)$ solely determines agent's risk preferences.

thus never sign the contract. Similarly, there is no incentive to set such β for $qdu > 1$ to hold: it brings no additional expected revenue (as e is already maximal), but would increase expected costs. Therefore, principal would only set such β that $0 \leq qdu \leq 1$.

So, principal is maximizing her expected utility

$$E(V(Pr)) = qe^*V(p - \alpha - \beta) + (1 - qe^*)V(-\alpha) \tag{14}$$

under agent's individual rationality constraint

$$qe^*u(\alpha + \beta) + (1 - qe^*)u(\alpha) - \frac{(e^*)^2}{2} \geq \bar{U}. \tag{15}$$

Combining (14) and (15) while substituting e^* from (13) yields the following utility maximization problem:

$$\begin{cases} q^2duV(p - \alpha - \beta) + (1 - q^2du)V(-\alpha) \to \max\limits_{\alpha,\, \beta}; \\ \text{s.t. } (q^2du)u(\alpha + \beta) + (1 - q^2du)u(\alpha) - \frac{(qdu)^2}{2} \geq \bar{U}, \end{cases} \tag{16}$$

which after some rearrangements becomes

$$\begin{cases} V(-\alpha) + q^2du(V(p - \alpha - \beta) - V(-\alpha)) \to \max\limits_{\alpha,\, \beta}; \\ \text{s.t. } u(\alpha) + \frac{1}{2}q^2(du)^2 \geq \bar{U}. \end{cases} \tag{17}$$

2.2a Model with risk-neutral principal and agent

In order to obtain the solution for the principal's problem we stated above, we would resort to our simplifying assumptions about utility functions made earlier in 2.1a. Then, $du = u(\alpha + \beta) - u(\alpha) = a(\alpha + \beta) - a(\alpha) = a\beta$, and thus (17) becomes the following:

$$\begin{cases} -\alpha + q^2a\beta(p - \beta) \to \max\limits_{\alpha,\, \beta}; \\ \text{s.t. } a\alpha + \frac{1}{2}q^2a^2\beta^2 \geq \bar{U}. \end{cases} \tag{18}$$

Note that again the individual rationality constraint in (18) is binding, and thus problem yields $\beta^* = p$. Under this level of bonus payment, agent will choose $e^* = aqp$, and from binding individual rationality constraint we obtain $\alpha^* = \frac{\bar{U}}{a} - \frac{1}{2}a(q\beta^*)^2 = \frac{\bar{U}}{a} - \frac{a(qp)^2}{2}$. Therefore, the resulting expected profit of the principal would be:

$$E(Pr) = \frac{a(qp)^2}{2} - \frac{\bar{U}}{a}. \tag{19}$$

However, there is one important caveat to be mentioned - throughout the analysis performed above, we assumed that the optimal level of the effort chosen is given by aqp, while in fact it should be bounded by zero from below and by unity from above. The lower bound is of no concern, as all three of the constants are positive - thus, optimal effort never hits zero. However, the upper bound might cause some additional doubts - it might be so that $aqp > 1$. In such a case, why would the principal pay a bonus of $\beta^* = p$, if it does not increase agent's effort - it seems that doing so creates additional costs with no additional revenues?

However attractive this intuition might be, it does not take into account that due to the binding nature of the individual rationality constraint, any increase in β^* is mirrored by a decrease in α^*. Moreover, as effort has already reached its maximal level, there is no need to increase expected compensation for the agent to offset additional cost of effort. So, in fact expected principal's profit becomes $E(Pr) = pq - \frac{\bar{U}+1/2}{a}$ iff $apq \geq 1$.

But before we jump to the comparison of principal's expected profits, we would briefly discuss changes in the contract structure as compared to section 2.1a. Before efforts were introduced into the model, contract structure in terms of the optimal bonus size was *irrelevant* for the outcome. However, now it is highly relevant - in fact, full risk transfer from principal to the agent is observed (assuming $apq \leq 1$). This is an expected consequence of the effort introduction - principal has to motivate the worker, and due to risk-neutrality assumptions, it is irrelevant which of the parties is exposed to the risk.

Finally, in order to compare the outcome for the principal obtained here with one where no endogenous efforts were present, we would plot expected profits for the principal on Figure 1.

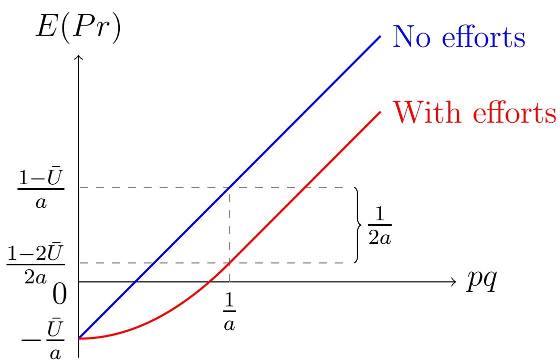

Figure 1: Expected profit with and without endogenous efforts in the model

This figure illustrates how the introduction of the unobservable efforts into the model changes the outcomes for the principal[16]. For $pq > \frac{1}{a}$, profits of the principal are the in both of the models - $\frac{1}{2a}$ only appears as the utility function of the agent has changed, and now agent has to be compensated for the efforts. In fact, $\frac{1}{2a}$ is exactly the monetary value of the unit efforts for the agent - additional compensation he requires in order to attain utility level of $\bar{U}$.

As for the $pq < \frac{1}{a}$ case, here the expected profits in two models differ by less than $\frac{1}{2a}$. For this particular case, it is more profitable for the principal to induce lower than maximal efforts from the agent, as additional expected revenues do not cover additional compensation required. Therefore, as principal is not entitled to pay for unit efforts, she can thus achieve expected profits

[16]Note that irrespective of whether the efforts are present in the model, agent will always have $E(U) = \bar{U}$.

higher than those from the model with no efforts less $\frac{1}{2a}$. Also note that all the parameters have the same effect on the expected profits as in section 2.1a[17].

2.3 Model with one agent and endogenous observable efforts

Alternatively, we could have assumed that principal is able to observe not only agent's success or failure, but also the efforts he makes. That would allow principal to implement an incentive scheme based not on value of y, but on the value of e, in order to enforce some desired level of effort, e^*:

$$w = \begin{cases} w^h, \text{ if } e = e^*; \\ w^l, \text{ if } e \neq e^*. \end{cases} \tag{20}$$

In order for e^* to be indeed chosen by the agent, two constraints must hold: individual rationality, (21), and incentive compatibility, (22):

$$U(w^h, e^*) \geq \bar{U}; \tag{21}$$

$$U(w^h, e^*) \geq U(w^l, e) \;\forall\; e \in [0, 1]. \tag{22}$$

Note that as $\frac{\partial U}{\partial e} < 0$, (22) is effectively $U(w^h, e^*) \geq U(w^l, 0)$. Moreover, as principal is able to set w^l low enough for this constraint to hold, she might lower it even further for $U(w^l, 0) \leq \bar{U}$ to be the case, and thus only IR constraint will be binding for principal.

Therefore, making the assumptions about utility function of the agent to be the same as in section 2.2, we can rewrite principal's problem to be as follows:

$$\begin{cases} qe^* V(p - w^h) + (1 - qe^*)V(-w^h) \to \max\limits_{w^h,\, e^*}; \\ \text{s.t. } u(w^h) - \frac{(e^*)^2}{2} \geq \bar{U}. \end{cases} \tag{23}$$

2.3a Model with risk-neutral principal and agent

In order to compare results under observable efforts with those obtained earlier, we make the same assumptions about risk-neutrality as in section 2.2a. Then, the problem stated above becomes

$$\begin{cases} qe^*(p - w^h) + (1 - qe^*)(-w^h) \to \max\limits_{w^h,\, e^*}; \\ \text{s.t. } aw^h - \frac{(e^*)^2}{2} \geq \bar{U}, \end{cases} \tag{24}$$

and since the constraint will be binding for principal to maximize profits, we would obtain the results identical to ones derived earlier in section 2.2a.

This occurs due to the fact that under bilateral risk-neutrality there are no costs of risk-transfer between agent and principal. Since under unobservable efforts principal has to link agent's

[17] In terms of effect sign, not effect magnitude.

compensation schedule to production outcome (in other words, set bonus payment $\beta > 0$) in order to motivate the agent. However, this bonus payment now leads to some of the risk being shifted towards the agent - and as agent is risk-neutral, there are no costs of it. However, if the agent was risk-averse, such risk-transfer would create costs for the principal in terms of expected profits - the phenomena called “cost of information”.

3 Two agent models of contracts under O-ring production function

3.1 Model with two agents and unobservable individual production results

Suppose that now principal requires two agents to produce the good. It means that the production function becomes $y = x_1 x_2$, where x_i stands for i-th agent individual output. Therefore, principal's profit is given by $Pr = py - w_1 - w_2 = px_1x_2 - w_1 - w_2$, where w_i is compensation for i-th agent. In this model we assume that the principal does not observe not only the effort levels of the agents, but also success or failure of individual assignments x_1 and x_2. Therefore, the only way for the principal to set bonuses is to condition them on the overall output, y. Thus, we assume that compensation for i-th agent has the same two-part form $w_i = \alpha_i + \beta_i y$ as previously.

The principal sets compensation schedule parameters α_1, α_2, β_1 and β_2 in order to maximize her expected payoff, given by

$$E(V(Pr)) = q_1e_1q_2e_2V(p - \beta_1 - \beta_2 - \alpha_1 - \alpha_2) + (1 - q_1e_1q_2e_2)V(-\alpha_1 - \alpha_2). \tag{25}$$

Utility function of agent i is given by $U_i = u(w_i) - \frac{e_i^2}{2}$, and therefore, each agent i maximizes his expected utility $E(U_i) = q_ie_iq_{-i}e_{-i}u(\alpha_i + \beta_i) + (1 - q_ie_iq_{-i}e_{-i})u(\alpha_i) - \frac{e_i^2}{2}$, with respect to e_i while considering e_{-i} to be fixed.

Individual FOC for agent i would be $\frac{\partial E(U_i)}{\partial e_i} = q_iq_{-i}e_{-i}u(\alpha_i + \beta_i) - q_iq_{-i}e_{-i}u(\alpha_i) - e_i = 0$, and thus optimal level of effort being

$$e_i^* = \begin{cases} 1, \text{ if } q_iq_{-i}e_{-i}du_i > 1; \\ q_iq_{-i}e_{-i}du_i, \text{ if } 0 \leq q_iq_{-i}e_{-i}du_i \leq 1; \\ 0, \text{ if } q_iq_{-i}e_{-i}du_i < 0, \end{cases} \tag{26}$$

where $du_i = u(\alpha_i + \beta_i) - u(\alpha_i)$ and represents a utility "premium" or differential caused by the bonus payment β_i added to compensation in case of successful production of output[18]. Since both agents solve their respective problems simultaneously, this solution represents best-response function of agent i to level of effort e_{-i} chosen by the second employee.

However, the resulting equilibrium heavily depends the slopes on the best-response functions, $q_1q_2du_1$ and $q_1q_2du_2$. If their product is less than unity, we would observe the situation as presented in case A) in Figure 2: both agents choose zero level of effort. If this product is greater than unity, there would be two potential equilibria in terms of efforts - (0,0) and one at which at least one of the agents extorts full effort (equal to unity). For non-zero equilibrium to be (1,1) it is necessary and sufficient for $q_1q_2du_1$ to be greater than unity for both agents 1 and 2 (see case B) in Figure 2).

[18]Note that the SOC for maximum holds here, as $\frac{\partial^2 E(U_i)}{\partial {e_i}^2} = -1 < 0$.

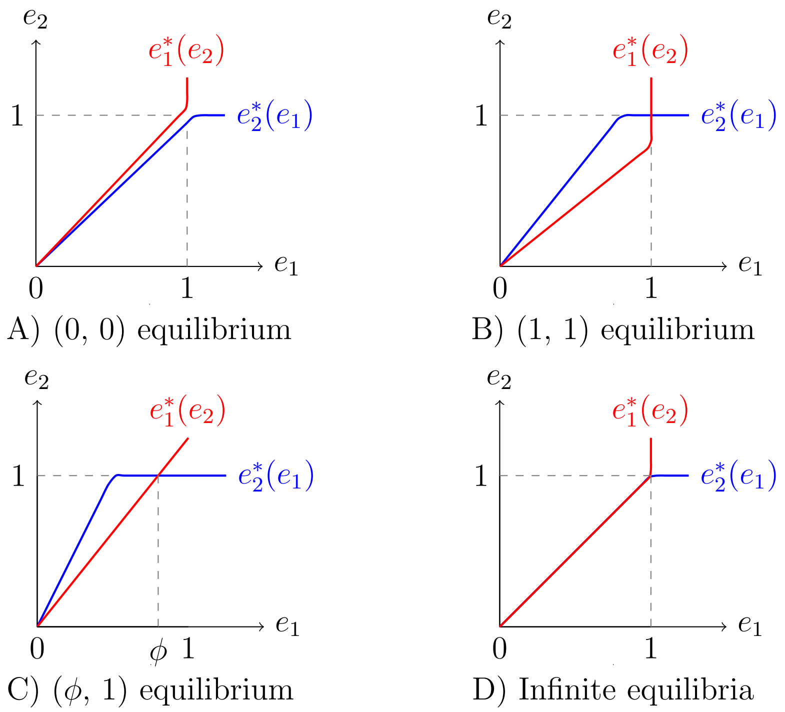

Figure 2: Equilibria for two-agent effort choice model

If for one of the agents this condition does not hold, the resulting equilibrium will be as in C) - agent with slope lower than unity will extort effort $0 < \phi < 1$. In fact, it would be exactly equal to the slope of the best-response function, $q_1 q_2 du_i$ for agent i.

Note, however, that only non-zero equilibrium is stable in cases B) and C)[19] - so we would consider this equilibrium to be the likely outcome of the game.

Finally, there also exists a case when product of the slopes of the best-response functions is equal exactly to 1 - then, infinite number of equilibria exists, and outcome of the game is indeterminate (illustrated by case D) in Figure 2).

So, there are three distinct type of equilibria the principal should keep in mind when designing the contract: (0,0), (1,1) and either (ϕ, 1) or (1, ϕ). The first one is not an option for the principal - as zero effort implies zero probability of producing the output, there is no incentive to offer non-zero wages to workers.

Thus, the principal considers those three potential alternatives, and chooses the most profitable of them. The only constraints she faces are the individual rationality constraints, given by

$$q_i e_i q_{-i} e_{-i} u(\alpha_i + \beta_i) + (1 - q_i e_i q_{-i} e_{-i}) u(\alpha_i) - \frac{e_i^2}{2} \geq \bar{U}_i \tag{27}$$

for agent i.

[19] This is not precisely true - in fact, (0,0) equilibrium is stable everywhere along the axes, but nowhere else.

3.1a Model with risk-neutral principal and agents

In this section will we will solve principal's maximization problem stated in the end of section 3.1 given the assumptions about risk-neutrality of the utility functions as in section 2.1a: namely, $V(Pr) = Pr$ and $u(w) = aw$. Therefore, principal maximizes her expected profits, now given by a modified expression (25):

$$E(Pr) = q_1 e_1 q_2 e_2 (p - \beta_1 - \beta_2) - \alpha_1 - \alpha_2. \tag{28}$$

The first option for the principal is to set set such du_1 and du_2 through her choice of β_1 and β_2 that (0,0) equilibrium will be implemented, but she will get zero revenues, and thus have no need in hiring agents at all - so, expected profits would be zero (or, in case principal decides to employ the agent, the expected profit would be equal to $E(Pr) = -\frac{\bar{U}_1}{a_1} - \frac{\bar{U}_2}{a_2}$).

The second option is to implement (1,1) equilibrium, what would require condition

$$q_i q_{-i} du_i = q_i q_{-i} a_i \beta_i > 1 \tag{29}$$

to be satisfied for both of the agents. Let us investigate this option in more detail.

Suppose that agents differ in three dimensions: skill q, reservation utility $\bar{U}$ and relative preference for income a. Then, using our assumption about utility function and substituting $e_1 = e_2 = 1$ into (27), we transform individual rationality constraint for agent i:

$$a_i \alpha_i + q_i q_{-i} a_i \beta_i - \frac{1}{2} \geq \bar{U}_i. \tag{30}$$

As the principal has no incentive to pay to provide utility higher than reservation level, (30) becomes equality and yields optimal level of α_i^* for each level of β_i:

$$\alpha_i^* = \frac{\bar{U}_i}{a_i} + \frac{1}{2a_i} - q_1 q_2 \beta_i. \tag{31}$$

Let us now use (31) in order to simplify the expected profit of the principal, given by (28):

$$E(Pr) = pq_1 q_2 - \frac{\bar{U}_1}{a_2} - \frac{1}{2a_1} - \frac{\bar{U}_2}{a_2} - \frac{1}{2a_2}. \tag{32}$$

Note that as it was previously demonstrated in section 2.2a, profits are no longer dependent on bonuses, given by β_1 and β_2. It means that principal is able to set them high enough for (29) to hold.

Finally, there is also **the third option** to consider, i.e. to implement case C) from Figure 2. Suppose that principal wants to achieve equilibrium $(\phi, 1)$, or for the first agent to have lower than maximal level of effort, namely equal to

$$\phi^* = q_1 q_2 du_1 = q_1 q_2 a_1 \beta_1, \tag{33}$$

as we know from section 3.1.

We use this in order to transform individual rationality constraint (27) and obtain optimal values for fixed part of the compensation schedule for both of the agents:

$$\alpha_1^* = \frac{\bar{U_1}}{a_1} - \frac{1}{2}a_1(q_1q_2\beta_1)^2; \tag{34}$$

$$\alpha_2^* = \frac{\bar{U_2}}{a_2} - a_2q_1^2q_2^2\beta_1\beta_2 + \frac{1}{2a_2}. \tag{35}$$

Then, (28) could be expressed as

$$E(Pr) = pa_1\beta_1q_1^2q_2^2 - \frac{1}{2}a_1q_1^2q_2^2\beta_2^2 - \frac{1}{2a_2} - \frac{\bar{U_1}}{a_1} - \frac{\bar{U_2}}{a_2}, \tag{36}$$

and principal maximizes it with respect to β_1. FOC yields $\beta_1^* = p$, which is exactly the same as it was in one agent model discussed previously in section 2.2a[20].

Note that this result implies $e_1^* = a_1q_1q_2p$, and if this expression is greater or equal to unity, we obtain exactly **second option** with $(1,1)$ equilibrium. So, if it is true for both agents 1 and 2, principal would choose the $(1,1)$ equilibrium. If it is true only for one of the agents, principal will extort less than maximal effort from the agent for whom this condition does not hold, thus obtaining expected profits of

$$E(Pr) = \frac{1}{2}p^2a_1q_1^2q_2^2 - \frac{1}{2a_2} - \frac{\bar{U_1}}{a_1} - \frac{\bar{U_2}}{a_2} \tag{37}$$

if $a_1q_1q_2p < 1$ and $a_2q_1q_2p > 1$.

The tricky case occurs when the condition $a_iq_1q_2p > 1$ is not satisfied for both of the agents. If the principal would offer each of them the bonus of $\beta_i = p$, he would in fact obtain $(0,0)$ equilibrium. Thus, she has an option to offer one of them high enough β in order to make the (29) hold for one of the agents, and offer $\beta = p$ to the second one, thus achieving the **third option** equilibrium. However, it is not clear which of the two agents should be motivated to extort the maximal effort.

Let us use (37) to obtain the expected profits for $(\phi, 1)$ equilibrium labeled $E_1(Pr)$ and for $(1, \phi)$ equilibrium labeled $E_2(Pr)$. Then,

$$E_1(Pr) - E_2(Pr) = \frac{1}{2}p^2a_1q_1^2q_2^2 - \frac{1}{2a_2} - (\frac{1}{2}p^2a_2q_1^2q_2^2 - \frac{1}{2a_1}) =$$

$$= \frac{1}{2}(a_1 - a_2)(p^2q_1^2q_2^2 - \frac{1}{a_1a_2}), \tag{38}$$

and as $p^2q_1^2q_2^2 - \frac{1}{a_1a_2} < 0$ in our case, it is more profitable to motivate the worker with highest a_i to extort maximal effort. This result shows that skill q_i and reservation utility $\bar{U_i}$ are irrelevant in determining bonuses - only relative importance a_i of the consumption matters - other parameters only determine fixed part of the contract α_i and size of the expected profits $E(Pr)$.

[20]Note that SOC also holds here, as $\frac{\partial^2 E(Pr)}{\partial \beta_2^2} = -a_1q_1^2q_2^2 < 0$.

However, principal will never offer such contracts, as her expected profits are for sure negative in this case (see $a_i p q_1 q_2 < 1$ for both agent implies that sum of the first two terms in (37) is negative[21]). Moreover, it is also true even under less strict condition $a_1 a_2 p^2 q_1^2 q_2^2 < 1$.

In order to summarize the result in this section, we would describe model result in terms of principal's payoff:

$$E(Pr) = \begin{cases} \max(pq_1q_2 - \frac{\bar{U}_1}{a_2} - \frac{1}{2a_1} - \frac{\bar{U}_2}{a_2} - \frac{1}{2a_2}, 0), \text{ if } q_i q_{-i} a_i p > 1 \forall i; \\ \max(\frac{1}{2}p^2 a_i q_i^2 q_{-i}^2 - \frac{1}{2a_{-i}} - \frac{\bar{U}_i}{a_i} - \frac{\bar{U}_{-i}}{a_{-i}}, 0), \text{ if } q_i q_{-i} a_i p < 1 \text{ and } q_i q_{-i} a_{-i} p > 1; \\ \max(-\frac{\bar{U}_1}{a_1} - \frac{\bar{U}_2}{a_2}, 0) \text{ if } q_i q_{-i} a_i p < 0 \forall i, \end{cases} \tag{39}$$

Note that irrespective of which of the cases in (39) is in place, both agents will end up exactly with reservation utility.

3.2 Model with two agents and observable individual production results

One of the most important parts of previous section was concerned with best-response analysis of agents' behavior. It was present due to the fact that action of one of them influenced actions of second, as principal determined compensation based on overall production result. However, if principal was able to observe whether the given agent has succeeded in her specific task, he would be able to set compensations tied to the value of x_i, not y: $w_i = \alpha_i + \beta_i x_i$.

Then, the individual agent's problem becomes exactly the same as one considered in section 2.2, and has exactly the same solution as in (13). However, the principal's problem is quite different, as now she has not two, but four potential outcomes in terms of payoff. Her expected utility is given by the following expression:

$$E(V(Pr)) = q_1e_1q_2e_2V(p - \alpha_1 - \alpha_2 - \beta_1 - \beta_2) + q_1e_1(1 - q_2e_2)V(-\alpha_1 - \alpha_2 - \beta_1) + \\ (1 - q_1e_1)q_2e_2V(-\alpha_1 - \alpha_2 - \beta_2) + (1 - q_1e_1)(1 - q_2e_2)V(-\alpha_1 - \alpha_2). \tag{40}$$

Then, the principal uses (13) to substitute into (40) and individual rationality constraints, given by (15) in order to obtain final form of principal's problem:

[21] For this conclusion to hold, besides the assumptions stated above we would also require non-negative reservation utilities $\bar{U}_1$ and $\bar{U}_2$. This assumption is quite reasonable, as negative reservation utility would imply that agents would agree to a negative wage while providing zero effort - they would for sure be better off under no employment at all.

$$
\begin{cases}
q_1^2 du_1 q_2^2 du_2 V(p-\alpha_1-\alpha_2-\beta_1-\beta_2) + q_1^2 du_1(1-q_2^2 du_2)V(-\alpha_1-\alpha_2-\beta_1)+ \\
(1-q_1^2 du_1)q_2^2 du_2 V(-\alpha_1-\alpha_2-\beta_2) + (1-q_1^2 du_1)(1-q_2^2 du_2)V(-\alpha_1-\alpha_2) \to \max\limits_{\alpha_1,\,\alpha_2,\,\beta_1,\,\beta_2}; \\
\text{s.t. } u_1(\alpha_1) + \frac{1}{2}q_1^2(du_1)^2 \geq \bar{U}_1, \\
\text{s.t. } u_2(\alpha_2) + \frac{1}{2}q_2^2(du_2)^2 \geq \bar{U}_2.
\end{cases}
\tag{41}
$$

3.2a Model with risk-neutral principal and agents

Once again, by assuming risk-neutrality of the utility functions as in section 2.1a, we can rewrite principal's expected utility stated above in (41) as follows:

$$
\begin{aligned}
E(V(Pr)) = q_1^2 a_1\beta_1 q_2^2 a_2\beta_2(p-\alpha_1-\alpha_2-\beta_1-\beta_2) + q_1^2 a_1\beta_1(1-q_2^2 a_2\beta_2)(-\alpha_1-\alpha_2-\beta_1)+ \\
(1-q_1^2 a_1\beta_1)q_2^2 a_2\beta_2(-\alpha_1-\alpha_2-\beta_2) + (1-q_1^2 a_1\beta_1)(1-q_2^2 a_2\beta_2)(-\alpha_1-\alpha_2),
\end{aligned}
\tag{42}
$$

which in turn could be further simplified into

$$
E(V(Pr)) = q_1^2 a_1\beta_1 q_2^2 a_2\beta_2 p - q_1^2 a_1\beta_1^2 - q_2^2 a_2\beta_2^2 - \alpha_1 - \alpha_2. \tag{43}
$$

As the individual rationality constraints are again binding, we can rewrite them as $\alpha_i = \frac{\bar{U}_i}{a_i} - \frac{1}{2}a_i q_i^2\beta_i^2$, we can use them and (43) in order to transform (41) into an unconstrained maximization problem of principal's expected profits with respect to β_1 and β_2:

$$
E(Pr) = q_1^2 q_2^2 a_1 a_2\beta_1\beta_2 p - \frac{1}{2}q_1^2 a_1\beta_1^2 - \frac{1}{2}q_2^2 a_2\beta_2^2 - \frac{\bar{U}_1}{a_1} - \frac{\bar{U}_2}{a_2} \to \max_{\beta_1,\,\beta_2}. \tag{44}
$$

The FOCs are as follows:

$$
\begin{cases}
q_1^2 a_1(q_2^2 a_2 p\beta_2 - \beta_1) = 0; \\
q_2^2 a_2(q_1^2 a_1 p\beta_1 - \beta_2) = 0.
\end{cases}
\tag{45}
$$

While the solution seems simple enough - (0,0), it is in fact not always so, as is revealed by the following Hessian matrix:

$$
H = \begin{pmatrix} -q_1^2 a_1 & q_1^2 a_1 q_2^2 a_2 p \\ q_2^2 a_2 q_1^2 a_1 p & -q_2^2 a_2 \end{pmatrix}, \tag{46}
$$

Note that (46) is only negative definite if $q_1^2 q_2^2 a_1 a_2 p^2 < 1$. In, (0,0) would be a maximum only if the production is not particularly profitable. In this case principal will not provide incentives for agents to extort any efforts. Note how closely it does resemble the conclusion made in section 3.1a: there principal would also set zero bonuses (and most likely abstain from production process, since reservation utilities are positive).

However, there also remains a case when (46) is indefinite. In fact, it might well be that even FOC would not hold - it might be beneficial to increase β_i's infinitely[22]. Unfortunately, we are not able to deduce the solution formally in this section. One way to overcome this problem could have been to assume homogeneous agents - but this would yield no meaningful benchmark to leverage against our results in section 3.1a, as not interim case $(\phi, 1)$ would be present.

3.3 Model with two agents, endogenous unobservable efforts and minimal effort requirement

In this section we expand our analysis by restricting the level of effort the agent has to extort - for instance, suppose there is some minimal level of effort k $(0 < k < 1)$ that the principal is able to enforce. We would also modify agents i utility function in order not to "punish" agents for the k first units of effort: $U_i = u(w_i) - \frac{(e_i-k)^2}{2}$. Therefore, each agent i maximizes his expected utility $E(U_i) = q_i e_i q_{-i} e_{-i} u(\alpha_i + \beta_i) + (1 - q_i e_i q_{-i} e_{-i}) u(\alpha_i) - \frac{(e_i-k)^2}{2}$, with respect to e_i while considering e_i to be fixed.

Individual FOC for agent i would be $\frac{\partial E(U_i)}{\partial e_i} = q_i q_{-i} e_{-i} u(\alpha_i + \beta_i) - q_i q_{-i} e_{-i} u(\alpha_i) - (e_i - k) = 0$, and thus optimal level of effort being

$$e_i^* = \begin{cases} 1, \text{ if } k + q_i q_{-i} e_{-i} du_i > 1; \\ k + q_i q_{-i} e_{-i} du_i, \text{ if } 0 \leq k + q_i q_{-i} e_{-i} du_i \leq 1; \\ k, \text{ if } k + q_i q_{-i} e_{-i} du_i < 0, \end{cases} \tag{47}$$

what is almost the same result we obtained in an earlier subsection[23].

However, the implications of this new best-response function are quite different, as is evident from Figure 3. First of all, there is no more room for (0,0) equilibrium - moreover, it is not even possible to have a (k, k) equilibrium[24]! Secondly, only one equilibrium is in place in each situation - there is no room for infinite number of equilibria as we saw in D) of Figure 2.

For game outcome to be like in A) of Figure 3, it is sufficient that for both agents' best-response functions to have their slopes, $q_1 q_2 du_i$, to be greater or equal to $1 - k$. If this condition does not hold for both of the agents (see case B) in Figure 3), we would obtain equilibrium with i-th agent's effort being $e_i^* = \frac{k(1+q_i q_{-i} du_i)}{(1-q_i^2 q_{-i}^2 du_i du_{-i})}$. Finally, in the last case C) one of the agents has best-response function slope less than $1 - k$, it would imply that this agent has equilibrium level of effort lower than 1 and equal to the one obtained in case B).

[22] Please note that while in fact it is not necessarily so, in fact there is an upper bound on efforts, which will limit the degree to which the principal would want to increase bonuses. After e_i reaches unity, increase in β_i is only a matter of reallocating the payment between fixed and bonus parts with no impact on neither of the agents.

[23] Note that the SOC for maximum holds here, as $\frac{\partial^2 E(U_i)}{\partial {e_i}^2} = -1 < 0$.

[24] As previously, we make an implicit assumption that utility differentials du_i are positive - or otherwise the only equilibrium present would be (k, k).

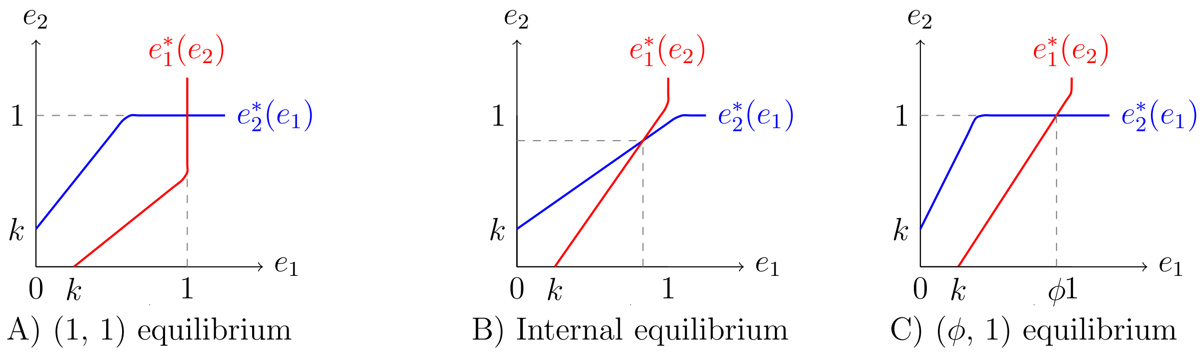

Figure 3: Equilibria for two-agent effort choice model with minimal effort requirement

We would not pursue the solution further, as it would almost the same as in case without minimal effort requirement. Purpose of this section was to establish one potential way to get rid of the "bad" equilibrium at zero observed in Figure 2. The important critique of our approach might be that we have simultaneously introduced two changes in the model to obtain results presented here - we incorporated k into cost of effort function and also imposed $e > k$ restriction. Let us discuss those two changes separately in a bit more detail.

On the one hand, if we were only to introduce minimal effort requirement into the cost of effort function, our plots in Figure 3 would not change at all: under non-negative du_i agent i would never lower his effort level below k, as it would only create additional costs of effort due to the quadratic form of the cost of effort function.

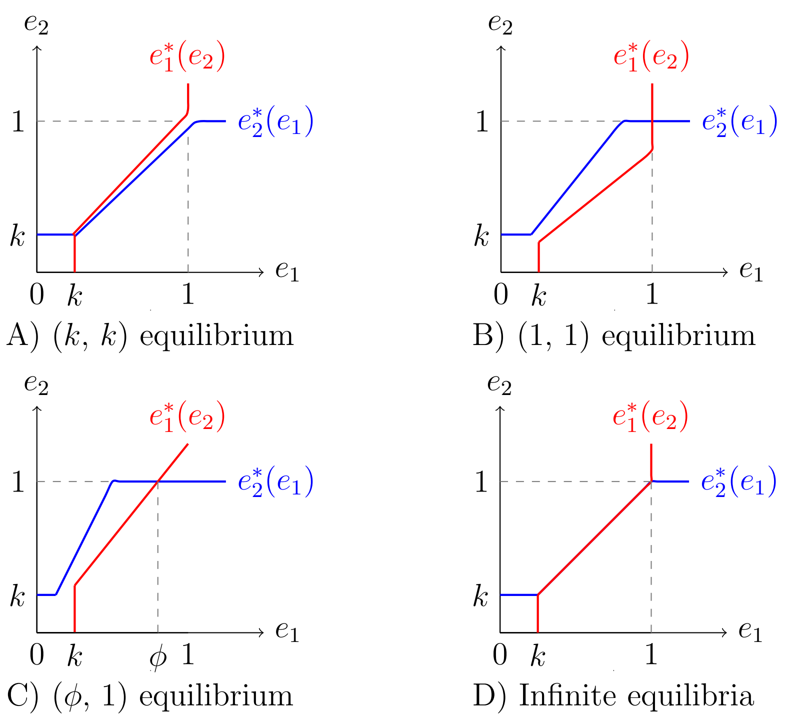

Figure 4: Equilibria for two-agent effort choice model with “floor” on effort

On the other hand, if we were to introduce only the "floor" on effort, the observed situation would resemble the one in Figure 4. All plots would be almost the same as in Figure 2, but the best-response curves would start from k. Note that the only qualitative consequence on the this change is that zero equilibrium is replaced with $(k,\ k)$, and thus paying relatively low bonuses does not fully crush principal's hopes for successful production, and thus her minimal expected revenues are $pk^2q_1q_2$ and not zero.

4 N agent models of contracts under O-ring production function

This part of the paper will generalize some of the results obtained for the two agent models presented above.

4.1 Model with N agents and endogenous unobservable efforts

This section will closely follow the analysis presented in section 3.1, but investigating the behaviour of agents and principal in the production of good that requires N agents. All of our assumptions will stay the same. Namely, the production function has the form of $y = \prod_{}^{N} x_i$, and therefore profit of the principal would be given by $Pr = p\prod^{N} x_i - \sum^{N} w_i$, as for each agent compensation schedule is given by $w_i = \alpha_i + \beta_i y$. Then, we can state that principal maximizes her expected payoff, given by:

$$E(V(Pr)) = (\prod^{N} q_i e_i)V(p - \sum^{N} \beta_i - \sum^{N} \alpha_i) + (1 - \prod^{N} q_i e_i)V(-\sum^{N} \alpha_i), \tag{48}$$

subject to individual rationality constraints for each agent i:

$$E(U_i) = \prod_{j}^{N} (q_j e_j) du_i + u(\alpha_i) - \frac{e_i^2}{2} \geq \bar{U}_i, \tag{49}$$

with respect to parameters α_1,...,α_N and β_1,...,β_N.

As for agents, each of them maximizes his expected utility with respect to his own level of effort, taking others' as given. The FOC would be

$$\frac{\partial E(U_i)}{e_i} = \prod_{j}^{N} q_j \prod_{j \neq i} e_j du_i - e_i = 0, \tag{50}$$

and thus best response function for agent i would be as follows:

$$e_i^* = \begin{cases} 1, \text{ if } (\prod_{j}^{N} q_j \prod_{j \neq i} e_j) du_i > 1; \\ (\prod_{j}^{N} q_j \prod_{j \neq i} e_j) du_i, \text{ if } 0 \leq (\prod_{j}^{N} q_j \prod_{j \neq i} e_j) du_i \leq 1; \\ 0, \text{ if } (\prod_{j}^{N} q_j \prod_{j \neq i} e_j) du_i < 0, \end{cases} \tag{51}$$

what is very similar to the one obtained for two-agent model in section 3.1[25].

[25]Note that even the SOC for maximum here is exactly the same, as $\frac{\partial^2 E(U_i)}{\partial {e_i}^2} = -1 < 0$.

However, looking for equilibria is not so straightforward here, as there are more than two agents here. One way to proceed could be to use second line in (51) to obtain the following relationship:

$$\prod_j^N e_j^* = \left(\prod_j^N q_j\right)^N \left(\prod_j^N e_j\right)^{N-1} \prod_j^N du_j, \tag{52}$$

which is in a sense the aggregate best response of all agents with respect to given product of efforts. This relationship is plotted on Figure 5, where $\theta = \left(\prod_j^N q_j\right)^N \prod_j^N du_j$ is coefficient before $\left(\prod_j^N e_j\right)^{N-1}$ in (52) and $\gamma = \left(\prod_j^N q_j\right)^{\frac{N}{N-2}} \left(\prod_j^N du_j\right)^{\frac{1}{N-2}}$ (note: $\frac{1}{\gamma}$ is the non-zero root of equation (52)).

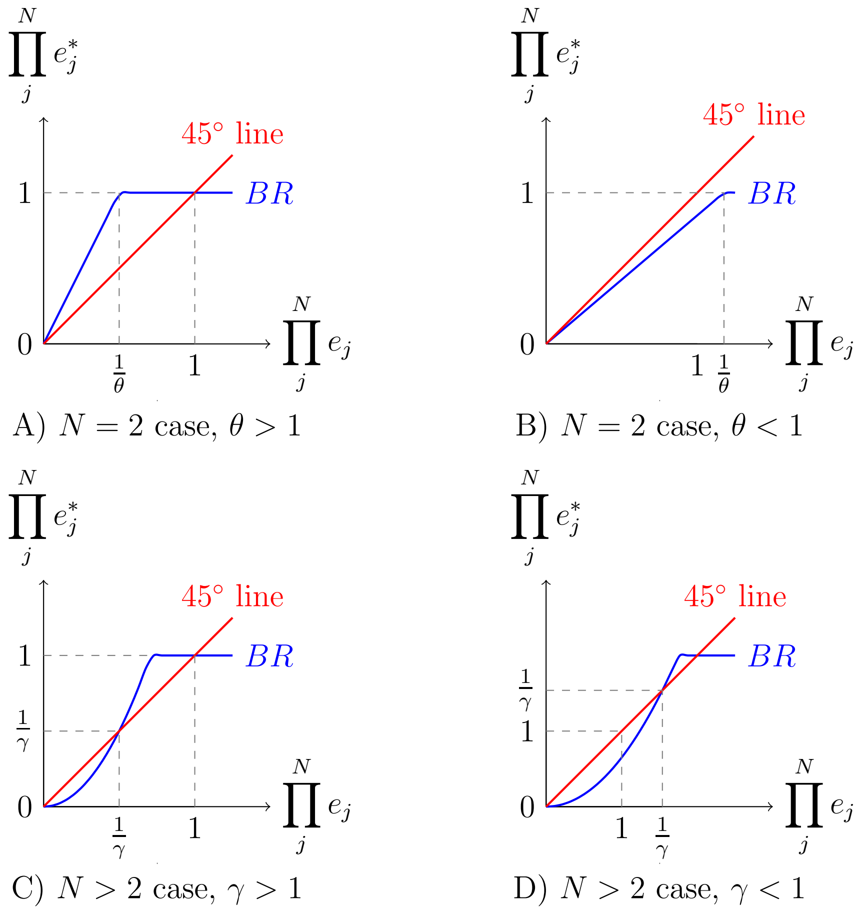

Figure 5: Aggregate best-response for N agents

We can consider that "equilibrium" will occur at some point where best-response function crosses 45° line. Let us first examine plots A) and B), which represent a special case of the problem with $N = 2$, which we explored in earlier section 3.1. Note that equation (52) under $N = 2$ has only one root - zero. However, the resulting equilibrium is determined by the relationship between slope of the aggregate best-response function, θ,and the 45° line. If best-response function is

steeper than the 45° line ($\theta > 1$), we would observe the case as illustrated by the plot A) in Figure 5: there are two potential equilibria, and only the right one (where $\prod_j^N e_j = 1$) is stable. On the contrary, if the best-response function is **flatter** than the 45° line ($\theta < 1$), we would obtain plot B), with zero being the stable equilibrium. Those results do match with one obtained earlier in section 3.1.

The $N > 2$ case is a bit more tricky, as now there non-zero root of equation (52), γ (note that it is for sure positive, given that all of its multiples are positive). If this root is a suitable value for efforts (namely, $\gamma > 1$), we would observe a case similar to plot C) in Figure 5. Note have additional equilibria at $\left(\prod_j^N e_j\right)^{N-1} = \frac{1}{\gamma}$ and at $\left(\prod_j^N e_j\right)^{N-1} = 1$. However, this intermediate equilibrium is unstable, while the two on the the boundaries are stable. Alternatively, γ could be greater than unity, and therefore there would be no equilibrium other than one at zero (see plot D) in Figure 5).

For sure, in case of $N > 2$ situation as observed in plot D) is of little interest to the principal - probability of successful production there is zero. Plot C), however, shows that this probability might well be at its maximal level (given by $\prod_j^N q_j$). Unlike the $N = 2$ case, there is no unique stable equilibrium, as both $\prod_j^N e_j = 0$ and $\prod_j^N e_j = 1$ are possible. There is no clear way to determine which equilibrium will be the outcome of the game between agents, but we might speculate that the answer to this question heavily depends on the "initial" point. If it is to the left from $\frac{1}{\gamma}$, we would expect agent arrive to the "bad" equilibrium, while it if is to the right of this point, it would be reasonable to expect a "good" one.

So, in this very simplistic setting, it would see that principal would be willing to shift $\frac{1}{\gamma}$ as close to zero as possible - and this claim implies greater γ would be viewed as beneficial to the principal[26]. What are the factors determining the value of γ? First of all, increase in any of the agent skill parameters, q_j helps to increase γ - so, principal would prefer skilled workers to unskilled, as

$$\frac{\partial \gamma}{\partial q_i} = \frac{N}{N-2} q_i^{\frac{2}{N-2}} \left(\prod_{j \neq i}^N q_j\right)^{\frac{N}{N-2}} \left(\prod_j^N du_j\right)^{\frac{1}{N-2}} > 0.$$

Also note that increased ability of one agent contributes more if all other agents are relatively skilled - so, ability complementarity is still present. The second way to increase γ would be to have greater du_i, utility differentials. This effect could be achieved either through greater bonus size β_i or through agent's greater preference for income (under our risk-neutrality assumptions it would be represented by a_i). Finally, there is also number of tasks (or agents), N, to consider.

[26]Please note that besides increasing the "likelihood" of "good" equilibrium, increasing γ also allows to switch form plot D) in Figure 5 to plot C) - thus guaranteing that "good" equilibrium is present at all.

The total effect of this parameter is ambiguous, but we can state that greater N increases relative importance of $\prod_j^N q_j$, while the effect on $\prod_j^N du_j$ is also ambiguous[27].

While this analysis of "aggregated" best-response function provides valuable insight, it does not account for restriction of efforts being less than unity at *individual* level. Consider the case with $N = 2$ again: our aggregated analysis does not show the case of plot C) in Figure 2, where one of the agents has effort level lower than unity. This happens due to the fact that γ does not account for values of slopes of individual best-response functions. In fact, we effectively forget that after some point some of the agents might not be able to increase their effort level (as would already reach unity), and thus the curve plotted on the Figure 5 would become flatter, as increased efforts would not have the same effect on individual efforts as before. This is a crucial flaw of the analysis performed above, and for sure limits our ability to investigate the model further on.

One way to overcome this constraint is to consider fully homogeneous agents that do not differ from each other in any way. Then, as principal would have no reason to give agents different compensation schedules, our analysis in Figure 5 would be fully accurate. Note, however, that this proposition does not lift up influence from N - $\gamma = q^{\frac{N^2}{N-2}} du^{\frac{N}{N-2}}$. The very first important conclusion to be made that now we are able to analyze the impact on "adding" one agent without considering that $\prod_j^N q_j$ being the same. Now, we clearly see that due to $0 < q < 1$ greater N decreases the "skill" component of γ, while effect on the "utility component" stays the same.

Unfortunately, we are not able to conduct our analysis in this section further: as our next step would be to assess how the principal sets compensation, we would need the expression for her expected profits. Unfortunately, we are not able to say whether the resulting equilibrium would be a "good" one or a "bad" one, as we do not have any indication of what might be the initial point. Even if we were to assume that the "good" equilibrium always prevails, we would end up principal paying bonuses just high enough for $\frac{1}{\gamma} < 1$ to be true. Another way to proceed might be to think of $1 - \frac{1}{\gamma}$ as a probability that a "good equilibrium" will be achieved. Then, probability of successful production would be $\xi = q^N(1 - \frac{1}{\gamma})$. We would briefly outline the principal's problem under this approach (and under assumptions of risk-neutrality) in the next section.

4.1a Model solution outline under risk-neutrality

The principal's objective function, terms of expected profits, would be as follows:

$$E(Pr) = \xi(p - N\beta) - N\alpha,$$

[27] Here we are assuming that those products remain constant, or that geometric average of q_j and du_j stays constant - note that that is not necessarily the case. The effects are due to the fact that we know for sure that each $0 < q_j < 1$, and we can not say the same about du_j.

or, under the IR constraint being $\alpha = \frac{\bar{U}}{a} + \frac{1}{2a} - \xi\beta$, given that agents would extort full effort in "good equilibrium", which occurs with probability ξ, it would have the following form:

$$E(Pr) = \xi(p - N\beta) - N(\frac{\bar{U}}{a} + \frac{1}{2a} - \xi\beta) = \xi p - \frac{N\bar{U}}{a} - \frac{N}{2a}. \tag{53}$$

After substituting

$$\xi = q^N(1 - \frac{1}{\gamma}) = q^N(1 - q^{-\frac{N^2}{N-2}}(a\beta)^{-\frac{N}{N-2}}) \tag{54}$$

into (53), we obtain the following problem:

$$\begin{cases} E(Pr) = (q^N - q^{-\frac{2N}{N-2}}(a\beta)^{-\frac{N}{N-2}}))p - \frac{N\bar{U}}{a} - \frac{N}{2a} \to \max\limits_{\beta}; \\ \text{s.t. } q^{-\frac{N^2}{N-2}}(a\beta)^{-\frac{N}{N-2}} < 1. \end{cases} \tag{55}$$

Note that the solution to this problem is fairly simple - $\beta \to \infty$. Why does it happen? There are several reasons for this outcome. Firstly, note that efforts are fixed at maximal level - that means that increased beta does not change costs of efforts for agents. Moreover, as agents are risk neutral and principal pays compensation exactly to satisfy individual rationality constraint, any increase in beta is mirrored by a decrease in fixed part of the compensation, salary α. As bonuses beta are costs to the principal who also pays fixed part of the compensation and she is also risk-neutral, increase in β does not change the amounts she pays to the agent. The only thing the beta impacts in terms of profits is the probability of the successful production, $\frac{1}{\gamma}$, which asymptotically approaches 1 as β approaches infinity - and that occurs due to the fact that we assumed that $\frac{1}{\gamma}$ could be viewed as the probability of "good equilibrium".

So, given the assumptions we made it is impossible to get reasonable results form the model as we state it - and the main cause of this is presence of multiple stable equilibria identified in previous section.

5 Conclusions

In this paper we have investigated how the optimal contracts would look like under the production function with different levels of complexity (with respect to number of agents, presence of efforts, their observability). The main contribution is the investigation of the two agent case, while efforts have been made to explore more complex cases. We also stated and emphasized the importance of minimal efforts level for the equilibrium efforts.

The important point to address here is the applicability of the this model in terms of explaining real-life phenomena. One example that might be relevant is the compensation systems in management consulting industry, as it does satisfy most of the assumptions made in this paper. The production (or project, using the industry jargon) outcome is indeed discrete - success could be defined either if there is a sell-on (another linked project that client is willing to pay for) or whether the consultants are paid at all[28]. Probability of project success for sure depends on both skills and efforts of the project team (consultants). However, even maximal efforts do not always ensure the project success, as there often arise unexpected issues. While the principal (partner on the project), who might be considered as residual claimant of the project proceeds. Moreover, since partners are rarely engaged in the actual project work due to tight schedules filled with meetings, they usually do not have an ability to monitor or enforce the the actual individual performance. Finally, it is not always clear even for the team members whether their colleagues are working hard (sometimes it is impossible to tell whether the specific task, especially connected with creative thinking, requires time T or time $4T$).

Since all that does closely follow the assumptions we have made, we can try to assess how closely the compensation systems in management consulting follow our conclusions. First of all, it is clear that bonuses do constitute a significant part of consultants income (from 10-20% in firms like PwC and EY to over 60% in McKinsey&Co, BCG and Oliver Wyman[29]). Even though the bonuses are usually not directly linked to a certain project (however, Strategy Partners Group and EY sometimes do that), consultants are given reviews that form their bonus and are directly linked to the project success. Moreover, it could be argued that such firms aim to hire the best talent not only for the sake of the project quality itself, but also due to the fact that better skills of consultants in team allow to pay less bonus to others - the effect in best-response functions we observed earlier. We can even state that this system works - it is not uncommon for consultants to pull a series of all-nighters, clearly extorting maximal efforts humanly possible.

After discussing the application of the paper, it is time to discuss the its limitations and potential further developments. First of all, some dynamics and repeated interactions would for sure be a reasonable next step, as it might at least partially solve the in-team moral hazard

[28]Clients in this industry are notorious for not paying for the projects they are not satisfied with the delivery of.

[29]Author would like to express his acknowledgements to the people in those firms who helped to gather such information: Olga Balusova, Vazgen Badalayan and Ilya Androsov.

problem. Another step to be made could concern the sequential production, as in original Kremer's paper. Also, a more accurate treatise of the N-agent case is required in order to make more clear conclusions. Finally, risk-preferences of the agents might be different for sure. Unfortunately, it was possible to obtain closed-form solutions for various functions (e.g. $u = aw^r$, $u = alnw$, $u = -e^{-rw}$ - probably the parametrization and numerical computations are required to proceed further.

Overall, we would conclude that this paper does contribute to the stream of literature that attempts to find optimal contracts in various settings - and we do it for such a very special and complex case as O-ring production functions.

6 References

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